Are all Subformulae Revelant to Prove Validity?

(1) ‘$(Q \supset R)$ & $(S \supset T)$’, ‘$(U \supset V)$ & $(W \supset X)$’, ‘$Q \vee U$’ ⊦ ‘$R \vee V$’

If we try a natural deduction to show the validity of (1) (cf. *Notes and Exercises*, p. 28, Ex. 6c):, it seems *prima facie* that we need $(Q \supset R)$, i.e. the first conjunct of the first premise, and $(U \supset V)$, i.e. the first conjunct of the second premise. But it seems that $(S \supset T)$ and $(W \supset X)$ are *not* required too. So, can we just ignore them? Yes.

Here is the natural deduction for (1) (see also *Notes and Exercises*, p. 57):

1	(1)	$(Q \supset R)$ & $(S \supset T)$	Premise
2	(2)	$(U \supset V)$ & $(W \supset X)$	Premise
3	(3)	$Q \vee U$	Premise
1	(4)	$Q \supset R$	1 &E
5	(5)	Q	Assumption
1, 5	(6)	R	4, 5 ⊃E
1, 5, 6	(7)	$R \vee V$	6, ∨I
2	(8)	$U \supset V$	2 &E
9	(9)	U	Assumption
2, 9	(10)	V	8, 9 ⊃E
2, 9, 10	(11)	$R \vee V$	10 ∨I
1, 2, 3	(12)	$R \vee V$	3, 5, 6, 7, 9, 10, 11 ∨E

Reasoning. Assuming Q at (5) we can (a) derive R, which is one of the disjuncts of the conclusion, by *modus ponens* (i.e. ⊃E), and (b) since Q is the first disjunct of the premise at (3), we can apply ∨E. (Using ∨E we can eliminate the disjunction $\phi \vee \psi$ if some proposition can be derived from both ϕ and ψ; see *Notes and Exercises*, p. 25.) At (7), we introduce V, as this is the second disjunct of the conclusion. (∨I allows us to add *any* proposition to ϕ.) At (9), the second disjunct U is assumed and used to derive V at (10). Again, at (11) a proposition is added by ∨I, which in this case is R, since it is the first disjunct of the conlusion. So, we have derived $R \vee V$ twice from both Q and U, and hence the disjunction $Q \vee U$ is fully used in the derivation and all assumptions discharged. But we have now derived $R \vee V$ without *using* $(S \supset T)$ and $(W \supset X)$. So be it.

Background. The first premise is a conjunction ϕ & ψ, which is true iff both conjuncts are true. So, ϕ is true, and ψ is true. This is why the &A/&E rule works (it preserves truth). Now, the question with regard to checking the validity of (1) is this: is it possible that a falsehood follows from either conjuncts? And: could the argument be invalid because of $(S \supset T)$ and $(W \supset X)$? No.

Since the conclusion includes R, we have to check the first conjunct, which involves R as atomic formula. But not the second conjunct, for it is not possible to derive R from $(S \supset T)$. So, since we could not derive R or V *at all* from $(S \supset T)$ and $(W \supset X)$, respectively, we could not derive a *falsehood* either.

But the disjunction $(Q \vee U)$ can be used to apply ∨E in connection with the two first conjuncts of the first two premises; it thus matters in deriving $(R \vee V)$.

To show that the argument is valid even with the omission, below is the tree for (1). Note that even here, $(S \supset T)$ and $(W \supset X)$ do not do any real work. As in the natural deduction, it seems that we could leave out the analyses of (6) and (8), because there are no atomic formulae that *could* yield an inconsistency. This is because *S*, *T*, *W*, and *X* occur only in these formulae. So, we could have analysed (7) after line (12), and thus reached full inconsistency at line (13). But the full analysis of the tree's root propositions is good practice.

(1)	$(Q \supset R) \& (S \supset T)$ √	Prem.
(2)	$(U \supset V) \& (W \supset X)$ √	Prem.
(3)	$Q \vee U$ √	Prem.
(4)	$\sim(R \vee V)$ √	Conc. (negated)
(5)	$Q \supset R$ √	1, &A
(6)	$S \supset T$ √	1, &A
(7)	$U \supset V$ √	2, &A
(8)	$W \supset X$ √	2, &A
(9)	$\sim R$	4, ~∨A
(10)	$\sim V$	4, ~∨A
(11)	Q U	3, ∨A
(12)	$\sim Q$ ⋆ R ⋆	5, ⊃A
(13)	$\sim S$ T	6, ⊃A
(14)	$\sim W$ X $\sim W$ X	8, ⊃A
(15)	~U ⋆ V ⋆ ~U ⋆ V ⋆ ~U ⋆ V ⋆ ~U ⋆ V ⋆	7, ⊃A

The same result could be demonstrated with a truth table, of course, where the question of *leaving out* certain wffs in the deduction does not arise at all.

Note. The example is from I. Copi (*Introduction to Logic*, Macmillan, New York, 1972, p. 293). He suggest to work with derived rules, such as the *constructive dilemma* (C. D.), which has this form: $(\phi \supset \psi) \& (\alpha \supset \beta), \phi \vee \alpha \vdash \psi \vee \beta$. Given this derived rule, the natural deduction becomes rather straightforward:

(1)	$(Q \supset R) \& (S \supset T)$	Premise
(2)	$(U \supset V) \& (W \supset X)$	Premise
(3)	$Q \vee U$	Premise
(4)	$Q \supset R$	1, &E
(5)	$U \supset V$	2, &E
(6)	$(Q \supset R) \& (U \supset V)$	4, 5, &I
(7)	$R \vee V$	3, 6, C. D.