Modal Logic

Roughly speaking, modal logic is the logic of *qualified* propositions. Modalities are expressions like ‘it is necessary that’, ‘it is possible that’, ‘it is obligatory that’, ‘it is permissible that’, ‘it is known that’, ‘it is believed that’, ‘it was true that’, or ‘it will be true that’. Narrow modal logic is about necessity and possibility; broader modal logic includes temporal, deontic, and doxastic propositions.

In 1918, C. I. Lewis wondered how ‘ϕ, $\phi \supset \psi \vdash \psi$’, i.e. *modus ponens* could work when it is possible that P is false? Answer: strict implication ‘⥽’ (the fishhook).

Definition: ϕ ⥽ $\psi = {\sim}\Diamond(\phi \,\&\, {\sim}\psi)$. The diamond ‘$\Diamond$’ stands for ‘possible’: ‘$\Diamond\phi$’ says that it is possible that ϕ (possibly ϕ; it could be that ϕ; it might be that ϕ). Likewise, ‘${\sim}\Diamond\phi$’ says that ϕ is impossible. Given ‘$\Diamond$’, we can also define ‘necessary’: ${\sim}\Diamond{\sim}\phi$. This is now usually expressed with the box ‘$\Box$’: $\Box\phi$, which says that it is necessary that ϕ (necessarily ϕ; it must be that ϕ). Equivalences: (a) $\Diamond\phi \equiv {\sim}\Box{\sim}\phi$; (b) $\Box\phi \equiv {\sim}\Diamond{\sim}\phi$; (c) ${\sim}\Diamond\phi \equiv \Box{\sim}\phi$; (d) ${\sim}\Box\phi \equiv \Diamond{\sim}\phi$.

Syntax and Semantics

Add the necessity operator ‘$\Box$’ and the possibility operator ‘$\Diamond$’ to PL or QL (QL=). Modal operators are *monadic* and have the same scope as the negation ‘$\sim$’.[1]

Note that ‘$\Box$’ and ‘$\Diamond$’ are *not* truth-functional: (a) ‘$\Diamond$’ is redundant; (b) and (d) ‘$\Diamond$’ generates a tautology or contradiction, (c) ‘$\Diamond$’ is ‘$\sim$’.

ϕ	$\Diamond\phi$			
	(a)	(b)	(c)	(d)
T	T	T	F	F
F	F	T	T	F

Since Kripke, modality is often put in terms of *possible worlds*, which is a complete counterfactual (or hypothetical) yet possible scenario, or simply a way things could be. E.g., there is a possible world where grass is blue. The possible world where grass is green is the *actual* world (usually ‘$w^\star$’). Think of the rows in a truth table as possible worlds.

Given this, (i) ‘$\Box\phi$’ is true iff ϕ is true in *all* possible worlds, and (ii) ‘$\Diamond\phi$’ is true iff ϕ is true in *at least one* possible world. Tautologies come out true in all possible worlds; contradictions in none; and contingent propositions in some. Example: in w, if P is true and Q is false, then $P \vee Q$ is true in w, but $P \supset Q$ is false in w. In $w^\star$ ($w^\star \neq w$), if P is false and Q is true, then $P \vee Q$ is true in $w^\star$ and $P \supset Q$ is true in $w^\star$. Hence truth is world-bound or relative (cf. Handout 14, intensionality).[2] Given this jargon, ‘$\Box$’ and ‘$\Diamond$’ turn into *quantifiers* over possible worlds.

1 Further operators: $\mathsf{P}\phi$ (ϕ was the case), $\mathsf{F}\phi$ (it will be the case that ϕ), e.g., ‘It will have been the case that ϕ’ can be formalised as $\mathsf{FP}\phi$, which in QL could be: $\exists t \exists t^\star(t_0 < t \,\&\, t^\star < t \,\&\, \phi(t^\star))$. Also: $\mathsf{K}\phi$ (an agent knows that ϕ), $\mathsf{B}\phi$ (an agent believes that ϕ), $\mathsf{O}\phi$ (it is obligatory that ϕ), and $\mathsf{P}\phi$ (it is permissible that ϕ); e.g., $\mathsf{O}\phi \equiv {\sim}\mathsf{P}{\sim}\phi$, which says that what we must do is something it should be impermissible not to do.

2 How is it possible to evaluate the truth of a modal proposition in some or all possible worlds? Modal semantics work with ‘models’, which include a set of possible worlds, some evaluative interpretation function, and an ‘accessibility relation’. World v is accessible from w iff v is possible *relative* to w (often $w^\star$). If a proposition is true at v but *could* not be true in w, this is because facts in v are impossible from the point of view of, and so inaccessible from, w. So, $\Box\phi$ is true at w so long as ϕ is true in all worlds that are accessible from w.

Examples (with liberal use of letters):

(1) Cameron might lose the next election: $\Diamond L$

(2) I drink tea if I must: $\Box T \supset T$

(3) If dogs could talk, they might tell jokes: $\Diamond T \supset \Diamond J$; compare: $\Diamond(T \supset J)$

(4) Hume could have been Welsh: $\Diamond Wh$

(5) Round squares are impossible: $\sim\Diamond\exists x(Rx \,\&\, Sx)$; compare: $\sim\Diamond\forall x(Rx \,\&\, Sx)$

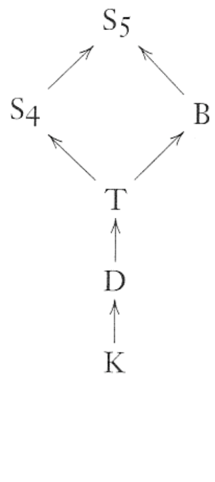
(6) Zombies are necessarily unconscious: $\forall x(Zx \supset \Box\sim Cx)$

The de re/de dicto *Distinction*

De re: $\exists x\Box Fix$ 'There is something such that it necessarily has a certain property'. A necessary or possible feature is predicted to an *object* (i.e. *res*). The *scope* of the modal operator is the thing and its property (but not the quantifier), but the variable is bound to the quantifier (not '$\Box$' or '$\Diamond$'). E.g., $\exists x\Diamond Fx$: 'There is something that may be on fire'.

De dicto: $\Box\exists xFx$ 'It is necessary that there is something that has a certain property'. This is a claim about a *statement* (i.e. a *dictum*). The modal operator's *scope* is the proposition, and the quantifier is inside the scope of the modal operator. E.g., $\Diamond\exists xFx$: 'There may be something on fire'.

Further Examples. (1) 'Nozick believes that taxation is theft.' (a) *De dicto*, Nozick has a certain belief about taxation. (b) *De re*, taxation is such that Nozick believes of it that it is theft. (2) 'The number of planets is necessarily odd' can be rendered as (c) $\Box\exists x(Nx \,\&\, \forall y(Ny \supset y = x) \,\&\, Ox)$; or (d) $\exists x(Nx \,\&\, \forall y(Ny \supset y = x) \,\&\, \Box Ox)$. (3) Berkeley claims that no sensible object exists unperceived, so that a thing's essence is to be perceived: something exists iff it is perceived.[3] Part of his argument is a claim about the conceivability of unperceived objects: "I am content to put the whole on this issue. If you can conceive it possible for any mixture or combination of qualities, or any sensible object whatsoever, to exist without the mind, then I will grant it actually to be so."[4] (To exist 'without' the mind means to exist unperceived.) But there is an ambiguity: (e) 'We can conceive the possibility that a sensible object exists unperceived': $\Diamond(\exists x)(Sx \,\&\, Ux)$, which is a *de dicto* claim; or (f) 'We can con-ceive that a sensible object possibly exists unperceived': $\exists x(Sx \,\&\, \Diamond Ux)$, which is *de re* conceivability.



There are several systems of modal logic that differ in strength, because of the axioms they accept. S5 is the strongest.[5] K is PL plus '$\Box$' and '$\Diamond$'. D is K plus $\Box\phi \supset \Diamond\phi$ (whatever is necessary is possible). T is D plus $\Box\phi \supset \phi$ (whatever is necessary is true).[6] B is T plus $\phi \supset \Box\Diamond\phi$ (what is the case is necessarily possible). S4 is T plus $\Box\phi \supset \Box\Box\phi$ (what is necessary is necessarily necessary). S5 is T plus $\Diamond\phi \supset \Box\Diamond\phi$ (what is possible is necessarily possible).

3 See, e.g., *Principles of Human Knowledge*, Part I, ¶3.

4 Berkeley, *Three Dialogues between Hylas and Philonous*, Dialogue I (p. 86 in the edition by J. Dancy, Oxford University Press, 1998).

5 Illustration: Sider, T. (2010). *Logic for Philosophy* (p. 144). Oxford: Oxford University Press.

6 In deontic logic, the equivalent $O\phi \supset \phi$ can be read as 'ought implies can'.