# Propositional Logic III

The semantical approach to PL: mapping truth values on PL formulae, which hence define the meaning of connectives. Two connectives deserve special attention, *viz*. the tautology and the contradiction.

| $\phi$ | $\psi$ | $\phi \top \psi$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | T |

| $\phi$ | $\psi$ | $\phi \perp \psi$ |
|---|---|---|
| T | T | F |
| T | F | F |
| F | T | F |
| F | F | F |

– A PL sentence $\phi$ is logically true iff $\phi$ is a *tautology*: $\phi$ comes out true under all possible evaluations (e.g., '$P \vee {\sim}P$').
– A PL sentence $\psi$ is logically false iff $\psi$ is a *contradiction*: $\psi$ comes out false under all possible evaluations. There is no situation where $\psi$ could be true; $\psi$ is an inconsistency (e.g., '$P$ & ${\sim}P$'; if we accept the law of the excluded middle).
– A PL sentence $\chi$ with at least one 'T' *and* at least one 'F' is contingent.

Tautologies are central for *validity*, i.e. it may not be the case that all the premises are true and the conclusion is false.

We can check the validity of an argument by the semantical method or by the syntactical method (i.e. natural deduction or propositional calculus, see Handout 7).

*Semantical Method.*
First Example: Sue drinks coffee or Jack is at school. If Jack is at school, Tom is at school too. But Tom is not at school. Therefore, Sue drinks coffee.
Premises: $P \vee Q$, $Q \supset R$, ${\sim}R$; Conclusion: $P$.

The argument is valid iff '$P \vee Q$', '$Q \supset R$', '${\sim}R$' $\vDash$ '$P$' ('$\vDash$' means 'tautologically entails', or 'the formulae before it impose truth on the formula after it'—this is why '$\vDash$' is called 'semantic turnstile').

It should not be possible to find a situation or evaluation (line of the truth table), where all premises are true and the conclusion false. (Hence: we can ignore true conclusions and false premises.)

| $P$ | $Q$ | $R$ | $P \vee Q$ | $Q \supset R$ | ${\sim}R$ | $P$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | F | T |
| T | T | F | T | F | T | T |
| T | F | T | T | T | F | T |
| T | F | F | T | T | T | T |
| F | T | T | **T** | **T** | F | **F** |
| F | T | F | **T** | F | **T** | **F** |
| F | F | T | F | **T** | F | **F** |
| F | F | F | F | **T** | **T** | **F** |

Second Example. If Jack wins, he buys a new tie. Jack does not win. Therefore, Jack does not buy a new tie. This argument is *invalid*. Here is why:

| $P$ | $Q$ | $P \supset Q$ | $\sim P$ | $\sim Q$ |
|---|---|---|---|---|
| T | T | **T** | F | **F** |
| T | F | F | F | T |
| F | T | **T** | **T** | **F** |
| F | F | T | T | T |

The above is hence a *formal fallacy*: denying the antecedens ($\phi \supset \psi$, $\sim\phi \not\models \sim\psi$). There is a related fallacy of affirming the consequent ($\phi \supset \psi$, $\psi \not\models \phi$). The corresponding *valid* versions are *modus (ponendo) ponens* ($\phi \supset \psi$, $\phi \models \psi$) and *modus (tollendo) tollens*
($\phi \supset \psi$, $\sim\psi \models \sim\phi$), respectively (see below).

*Linking* '⊨' *(Logical Implication) and* '⊃' *(Material Implication)*

There is an important connection between the PL-connective '⊃' and the meta-linguistic sign '⊨', which expresses a relation. Consider $\Gamma \models \phi$, which says that the conclusion $\phi$ logically follows from the set $\Gamma$ of all the premises: $\{\psi_1, \psi_2, \ldots, \psi_n\} \models \phi$. So, if the premises are true, the conclusion is true too. This is the definition of validity.

This is equivalent to saying that '$(\psi \& \psi_2 \& \ldots \& \psi_n) \supset \phi$' is a tautology: $\Gamma \models \phi$ iff $\models (\Gamma \supset \phi)$, which reads: the argument from $\Gamma$ to $\phi$ is tauologically valid iff the corresponding material implication ($\Gamma \supset \phi$) is a tautology, or if $\Gamma$ tautologically entails $\phi$.[1]

*Tautological entailment*. (a) Validity again: no possible evaluation of the premises that make $\Gamma$ true but $\phi$ false; (b) Definition of '⊃': every evaluation of the atomic propositions that appears both in $\Gamma$ and $\phi$ makes ($\Gamma \supset \phi$) true.[2] Here is proof:

*Modus pollens*

| $P$ | $Q$ | (($P$ | ⊃ | $Q$) | & | $P$) | ⊃ | $Q$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | **T** | T |
| T | F | T | F | F | F | T | **T** | F |
| F | T | F | T | T | F | F | **T** | T |
| F | F | F | T | F | F | F | **T** | F |

*Modus tollens*

| $P$ | $Q$ | (($P$ | ⊃ | $Q$) | & | $\sim Q$) | ⊃ | $\sim P$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | F | F | **T** | F |
| T | F | T | F | F | F | T | **T** | F |
| F | T | F | T | T | F | F | **T** | T |
| F | F | F | T | F | T | T | **T** | T |

*Problem*

This method does not work (in practice) for arguments like these:

– $(P \& \sim Q)$, $(P_{22} \vee Q_{13})$, $(\sim P_{41} \& \sim Q_7) \therefore R$, or
– $(P \supset Q)$, $R$, $(\sim S \vee (P \& T))$, $\sim((U \equiv W) \vee \sim Q) \therefore \sim P$,

which yield truth tables with $2^7 = 128$ lines.

1 ⊨ $\phi$ says that $\phi$ logically follows from an empty set of premises and is hence unconditionally true; e.g., ⊨ $(\phi \& \phi)$.

2 For discussion, see, e.g., Smith, P. (2003). *Formal Logic*. Cambridge: CUP (ch. 14.6).