Propositional Logic IV

"My starting point was this: The formalization of logical resoning, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction practically used in mathematical proofs. Although this yields considerable formal advantages in return, I intended first of all to set up a formal system which comes as close as possible to actual reasoning. The result was a '*calculus of natural deduction*'".[1]

Syntactical Method

An argument is valid if we can show that no falsehood derives from applying a sequence of truth-preserving inferences. Such a method is 'natural deduction'.

As noted, the double turnstile ('⊨') says that whenever sentences $\psi_1, \psi_2, \ldots, \psi_n$ are true, then ϕ is true too. This alternative method uses the single turnstile ('⊢'), which says that a sentence ϕ can be *derived* from sentences $\psi_1, \psi_2, \ldots, \psi_n$ using the rules of deduction. So, $\Gamma \vdash \phi$ says that a conclusion ϕ can be *deduced* from the set of sentences in Γ (i.e. the premises); or that ϕ can be *proven* from sentences in Γ. The syntactic method is thus a proof system. The deduction of ϕ is valid because each step in the chain of derivations is valid.

The Rules

- *Assumptions*: ϕ is assumed if underived; the assumption of ϕ is proof of ϕ. Any assumptions must be 'discharged' before (must not appear) in the conclusion.
- *Conjunction Introduction* (&I): go from ϕ and ψ to (ϕ & ψ). To prove (ϕ & ψ), first prove ϕ and ψ separately and the use &I. Justified by truth table. Aka: *Adjunction*.

$$\frac{\phi \quad \psi}{\phi \,\&\, \psi}$$

- *Conjunction Elimination* (&E): go from (ϕ & ψ) to ϕ; or go from ϕ & ψ to ψ. To prove ϕ, try proving (ϕ & ψ) and use &E. Justified by truth table: ϕ & ψ is true iff ϕ is true *and* ψ is true. Aka: *Simplification*.

$$\frac{\phi \,\&\, \psi}{\phi}$$

- *Disjunction Introduction* (∨I): go from ϕ to ($\phi \vee \psi$) or from ψ to ($\phi \vee \psi$). Justified by truth table. Aka: *Addition*.

$$\frac{\begin{matrix}\vdots\\ \phi\end{matrix}}{\phi \vee \psi} \qquad \frac{\begin{matrix}\vdots\\ \psi\end{matrix}}{\phi \vee \psi}$$

- *Disjunction Elimination* (∨E): if $\phi \vdash \chi$ and $\psi \vdash \chi$, then go from ($\phi \vee \psi$) to χ. If χ can be proven from both disjuncts, the disjunction can be eliminated (cf. truth table for '∨').

$$\frac{\begin{matrix} \\ \vdots \\ \phi \vee \psi\end{matrix} \quad \begin{matrix}[\phi]\\ \vdots\\ \chi\end{matrix} \quad \begin{matrix}[\psi]\\ \vdots\\ \chi\end{matrix}}{\chi}$$

1 Gentzen, G. (1935). Untersuchungen über das logische Schließen I. *Mathematische Zeitschrift*, *39*, 176–210 (p. 176; transl. P.W.) Jaskowski independently developed a similar calculus.

- *Negation Introduction* (~I): if $\phi \vdash (\psi, \sim\psi)$, infer $\sim\phi$ (i.e. *reductio ad absurdum*). (The square brackets indicate discharged assumptions, i.e. auxiliary premises that are introduced and later discarded.)

$$\frac{\begin{matrix}[\phi] & [\phi] \\ \vdots & \vdots \\ \psi & \sim\psi\end{matrix}}{\sim\phi}$$

- *Negation Elimination* (~E): go from $\sim\sim\phi$ to ϕ.
- *Implication Introduction* (⊃I): if $\phi \vdash \psi$, go from ψ to $(\phi \supset \psi)$. Justification: truth table for '⊃': $(\phi \supset \psi)$ is true if ψ is true (cf. paradoxes of the implication). Alternatively: given the definition of '⊃', it is not possible that ϕ is true and ψ is false. But if ψ is deduced from ϕ, and deduction preserves truth, ψ cannot be false. Aka: *Conditional Proof.*

$$\frac{\begin{matrix}[\phi] \\ \vdots \\ \psi\end{matrix}}{\phi \supset \psi}$$

- *Implication Elimination* (⊃E): go from $(\phi \supset \psi)$ and ϕ to ψ (i.e. *modus ponens*).

$$\frac{\begin{matrix}\vdots & \vdots \\ \phi & \phi \supset \psi\end{matrix}}{\psi}$$

A Selection of Further Rules

Disjunctive Syllogism: $\phi \vee \psi, \sim\phi \vdash \psi$ or $\phi \vee \psi, \sim\psi \vdash \phi$

Conjunctive Syllogism: $\sim(\phi \,\&\, \psi), \phi \vdash \sim\psi$, or $\sim(\phi \,\&\, \psi), \psi \vdash \sim\phi$

Hypothetical Syllogism: $(\phi \supset \psi), (\psi \supset \chi) \vdash (\phi \supset \chi)$

Modus Tollens: $\phi \supset \psi, \sim\psi \vdash \sim\phi$

Dilemmata, e.g., Constructive Dilemma, $\phi \supset \psi, \chi \supset \xi, \phi \vee \chi \vdash \psi \vee \xi$

Logical equivalences, e.g., Contraposition, $\phi \supset \psi \equiv \sim\psi \supset \sim\phi$

How Natural Deduction Works

First Example.[2] '$P \supset Q$', '$\sim Q$' ⊢ '$\sim P$'

1[3]	(1)	$P \supset Q$	Premise
2	(2)	$\sim Q$	Premise
3	(3)	P	Assumption[4]
1, 3	(4)	Q	1, 3 ⊃E
1, 2, 3	(5)	$Q \,\&\, \sim Q$	2, 4 &I
1, 2	(6)	$\sim P$	3, 5, ~I

Second Example. '$(P \,\&\, Q) \supset R$', '$S \supset P$', 'Q' ⊢ '$S \supset R$'

1	(1)	$(P \,\&\, Q) \supset R$	Premise
2	(2)	$S \supset P$	Premise
3	(3)	Q	Premise
4	(4)	S	Assumption
2, 4	(5)	P	2, 4 ⊃E
2, 3, 4	(6)	$P \,\&\, Q$	3, 5 &I
1, 2, 3, 4	(7)	R	1, 6 ⊃E
1, 2, 3	(8)	$S \supset R$	4, 7 ⊃I

2 The presentation of natural deductions varies.

3 The first column lists *dependencies*, the second is a running *numbering*, the third lists the PL *sentences*, the fourth lists the *justifications* for the deductions and the *rules* used.

4 Given the rule of assumptions, any proposition can be assumed, as long as they are 'discharged' *prior* to the conclusion. (They may not appear in first column of the last line.) In the present example, P is discharged when ~I is applied at line (6).