Practising Trees

(1) $\sim(P \vee \sim Q) \,\&\, R,\ Q \supset R,\ \vDash\ \sim P \,\&\, R$
Note. This example is different from the original worksheet. See the Additional Note on the puzzle.

First Option: 1–2–3

(1)	$\sim(P \vee Q) \,\&\, R$	√	Premise
(2)	$Q \supset R$	√	Premise
(3)	$\sim(\sim P \,\&\, R)$	√	Negated Conclusion
(4)	$\sim(P \vee \sim Q)$	√	1, &A
(5)	R		1, &A
(6)	$\sim P$		4, ~∨A
(7)	$\sim\sim Q$		4, ~∨A
(8)	$\sim Q$ * / R		2, ⊃A
(9)	(under R:) $\sim\sim P$ / $\sim R$		3, ~&A
(10)	* *		

Verdict. The first tree is quite lean and simple, and so very elegant.

Second Option: 1–3–2

(1)	$\sim(P \vee \sim Q) \,\&\, R$	√	Premise
(2)	$Q \supset R$	√	Premise
(3)	$\sim(\sim P \,\&\, R)$	√	Negated Conclusion
(4)	$\sim(P \vee \sim Q)$	√	1, &A
(5)	R		1, &A
(6)	$\sim P$		4, ~∨A
(7)	$\sim\sim Q$		4, ~∨A
(8)	$\sim\sim P$ * / $\sim R$		3, ~&A
(9)	(under $\sim R$:) $\sim Q$ * / R *		2, ⊃A

Verdict. The second tree closes just as swiftly. As long as the non-branching rule is applied first, the following sequence makes no difference. Very elegant too.

Third Option: 2–1–3

(1)	$\sim(P \vee \sim Q) \,\&\, R$ √			Premise
(2)	$Q \supset R$ √			Premise
(3)	$\sim(\sim P \,\&\, R)$ √			Negated Conclusion
(4)	$\sim Q$	R		2, ⊃A
(5)	$\sim(P \vee \sim Q)$	$\sim(P \vee \sim Q)$		1, &A
(6)	R	R		1, &A
(7)	$\sim P$	$\sim P$		5, ~∨A
(8)	$\sim\sim Q$	$\sim\sim Q$		5, ~∨A
	*			
(9)		$\sim\sim P$	$\sim R$	3, ~&A
		*	*	

Verdict. Not quite as elegant as the previous trees; a little clunky.

Fourth Option: 2–3–1

(1)	$\sim(P \vee \sim Q) \,\&\, R$ √				Premise
(2)	$Q \supset R$ √				Premise
(3)	$\sim(\sim P \,\&\, R)$ √				Negated Conclusion
(4)	$\sim Q$		R		2, ⊃A
(5)	$\sim\sim P$	$\sim R$	$\sim\sim P$	$\sim R$	3, ~&A
(7)	$\sim(P \vee \sim Q)$	$\sim(P \vee \sim Q)$	$\sim(P \vee \sim Q)$	*	1, &A
(8)	R	R	R		1, &A
(9)	$\sim P$	$\sim P$	$\sim P$		7, ~∨A
(10)	$\sim\sim Q$	$\sim\sim Q$	$\sim\sim Q$		7, ~∨A
	*	*	*		

Verdict. The fourth tree is not elegant, because we have to work through three of the four branches, and the third branch from left closes only just about.

Fifth Option: 3–1–2

(1)		$\sim(P \vee \sim Q) \,\&\, R$			√	Premise
(2)		$Q \supset R$			√	Premise
(3)		$\sim(\sim P \,\&\, R)$			√	Negated Conclusion
(4)	$\sim\sim P$		$\sim R$			3, ~&A
(5)	$\sim(P \vee \sim Q)$		$\sim(P \vee \sim Q)$			1, &A
(6)	R		R			1, &A
(7)	$\sim P$		$\sim P$			5, ~∨A
(8)	$\sim\sim Q$		$\sim\sim Q$			5, ~∨A
(9)	$\sim Q$	R	$\sim Q$	R		2, ⊃A
	*	* [?]	*	*		

Verdict. The fifth tree is quite ugly. We must analyse (2) on every branch, which leaves the one second to the left 'open'. We discussed this 'puzzle': since there is an inconsistency on the branch *before* line 9, all following branches close. A better tree seeks to avoid this.

Sixth Option: 3–2–1

(1)		$\sim(P \vee \sim Q) \,\&\, R$			√	Premise
(2)		$Q \supset R$			√	Premise
(3)		$\sim(\sim P \,\&\, R)$			√	Negated Conclusion
(4)	$\sim\sim P$		$\sim R$			3, ~&A
(5)	P					4, ~~A
(6)	$\sim Q$	R	$\sim Q$	R		2, ⊃A
(7)	$\sim(P \vee \sim Q)$	$\sim(P \vee \sim Q)$	$\sim(P \vee \sim Q)$	*		1, &A
(8)	R	R	R			1, &A
(9)	$\sim P$	$\sim P$	$\sim P$			7, ~∨A
(10)	$\sim\sim Q$	$\sim\sim Q$	$\sim\sim Q$			7, ~∨A
	*	*	*			

Verdict. The last tree is just too much work.

(2) $P \vee Q, \sim(P \,\&\, \sim R) \vDash Q \vee R$

The least elegant sequence: 1–2–3

(1)	$P \vee Q$	√				P.
(2)	$\sim(P \,\&\, \sim R)$					P.
(3)	$\sim(Q \vee R)$					N. C.
(4)	P			Q		1, ∨A
(5)	$\sim P$	$\sim\sim R$		$\sim P$	$\sim\sim R$	2, ~&A
(6)	*	$\sim Q$		$\sim Q$	$\sim Q$	3, ~∨A
(7)		$\sim R$		$\sim R$	$\sim R$	3, ~∨A
		*		*	*	

The most elegant sequence: 3–1–2 (and also 3–2–1)

(1)	$P \vee Q$	√		P.
(2)	$\sim(P \,\&\, \sim R)$	√		P.
(3)	$\sim(Q \vee R)$	√		N. C.
(4)	$\sim Q$			3, ~∨A
(5)	$\sim R$			3, ~∨A
(6)	P		Q	1, ∨A
			*	
(7)	$\sim P$	$\sim\sim R$		2, ~&A
	*	*		

Note. All sequences ought to have seven lines.

(3) $(P \,\&\, Q) \supset R \vDash (P \supset R) \vee (Q \supset R)$

(1)		$(P \,\&\, Q) \supset R$			P.
(2)		$\sim((P \supset R) \vee (Q \supset R))$		√	N. C.
(3)		$\sim(P \supset R)$		√	2, $\sim\vee$A
(4)		$\sim(Q \supset R)$		√	2, $\sim\vee$A
(5)		P			3, $\sim\supset$A
(6)		$\sim R$			3, $\sim\supset$A
(7)		Q			4, $\sim\supset$A
(8)		$\sim R$			4, $\sim\supset$A
(9)	$\sim(P \,\&\, Q)$		R *		1, $\supset$A
(10)	$\sim P$ * $\sim Q$ *				9, $\sim\&$A

Syntactic Method: Natural Deduction

(1) "If we read Plato, we become virtuous. If we become virtuous, we become happy. If we read Nietzsche, we become unhappy. But we read both Plato and Nietzsche. So, we become both happy and unhappy."

Sequent: $P \supset Q, Q \supset R, S \supset \sim R, P \,\&\, S \vdash R \,\&\, \sim R$

For the rules, see Reader pp. 32–3. The argument is valid, because it is possible to deduce the conclusion from the premises.

Suppes Style:

1	(1)	$P \supset Q$	P.
2	(2)	$Q \supset R$	P.
3	(3)	$S \supset \sim R$	P.
4	(4)	$P \,\&\, S$	P. / $R \,\&\, \sim R$
4	(5)	S	4, &E
3, 4	(6)	$\sim R$	3, 5, $\supset$E
4	(7)	P	4, &E
1, 4	(8)	Q	1, 7, $\supset$E
1, 2, 4	(9)	R	2, 8, $\supset$E
1, 2, 3, 4	(10)	$R \,\&\, \sim R$	6, 9, &I

Gentzen Style:

$$\dfrac{\dfrac{\dfrac{\dfrac{P \,\&\, S}{P} \qquad P \supset Q}{Q} \qquad Q \supset R}{R} \qquad \dfrac{\dfrac{P \,\&\, S}{S} \qquad S \supset {\sim}R}{{\sim}R}}{R \,\&\, {\sim}R}$$

(2) "Kant reads Hume. If so, then he wakes from his dogmatic slumbers. If Kant reads Hume, then he wakes from his dogmatic slumbers only if transcendental philosophy begins. So, Kant read Hume and transcendental philosophy begins."

Sequent: $P, P \supset Q, P \supset (Q \supset R) \vdash P \,\&\, R$

Suppes Style:

1	(1)	P	P.
2	(2)	$P \supset Q$	P.
3	(3)	$P \supset (Q \supset R)$	P. / $P \,\&\, R$
1, 2	(4)	Q	1, 2, ⊃E
1, 3	(5)	$Q \supset R$	1, 3, ⊃E
1, 2, 3	(6)	R	4, 5, ⊃E
1, 2, 3	(7)	$P \,\&\, R$	1, 6, &I

Line of thought. The last line will introduce &. So we need to get P and R out of the premises. P is actually a premise, so we do not need to derive anything. R is in line 3: in order to deduce it, we need Q, and in order to derive Q, we need to to eliminate ⊃ at line 2. We can do this by applying ⊃E to lines 1 and 2, which is a straightforward *modus ponens*. Then we extract $Q \supset R$ in line 3 by another *modus ponens*. Then we use Q to apply ⊃E once again to derive R. Finally, we can put P and R together by &I.

Gentzen Style:

$$\dfrac{\dfrac{\dfrac{P \qquad P \supset Q}{Q} \qquad \dfrac{P \qquad P \supset (Q \supset R)}{Q \supset R}}{R}}{P \,\&\, R}$$

(3) ~~(*P* & *Q*), ((*P* & *R*) ∨ *Q*) ⊃ *S* ⊢ *P* & *S*

Suppes Style:

1	(1)	~~(*P* & *Q*)	P.
2	(2)	((*P* & *R*) ∨ *Q*) ⊃ *S*	P. / *P* & *S*
1	(3)	*P* & *Q*	1, ~~E
1	(4)	*Q*	3, &E
1	(5)	(*P* & *R*) ∨ *Q*	4, ∨I
1, 2	(6)	*S*	2, 5, ⊃E
1	(7)	*P*	3, &E
1, 2	(8)	*P* & *S*	6, 7, &I

Gentzen Style:

$$\dfrac{\dfrac{\dfrac{{\sim}{\sim}(P \,\&\, Q)}{P \,\&\, Q}}{P} \qquad \dfrac{\dfrac{\dfrac{\dfrac{{\sim}{\sim}(P \,\&\, Q)}{P \,\&\, Q}}{Q}}{(P \,\&\, R) \vee Q} \quad ((P \,\&\, R) \vee Q) \supset S}{S}}{P \,\&\, S}$$

(4) *P* ⊃ (*Q* & *R*), ~*Q* ⊢ ~*P*

Suppes Style:

1	(1)	*P* ⊃ (*Q* & *R*)	P.
2	(2)	~*Q*	P. / ~*P*
3	(3)	*P*	Assumption
1, 3	(4)	*Q* & *R*	1, 3, ⊃E
1, 3	(5)	*Q*	4, &E
1, 2, 3	(6)	*Q* & ~*Q*	2, 5, &I
1, 2	(7)	~*P*	3–6, ~I

Gentzen Style:

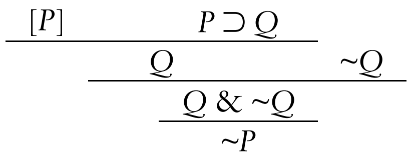
$$\dfrac{\dfrac{\dfrac{\dfrac{[P] \qquad P \supset (Q \,\&\, R)}{Q \,\&\, R}}{Q} \qquad {\sim}Q}{Q \,\&\, {\sim}Q}}{{\sim}P}$$

(5) $P \supset Q, {\sim}Q \vdash {\sim}P$ (see Reader p. 33)

Natural Deduction à la Suppes:

1	(1)	$P \supset Q$	Premise
2	(2)	$\sim Q$	Premise
3	(3)	P	Assumption
1, 3	(4)	Q	1, 3 ⊃E
1, 2, 3	(5)	$Q \& \sim Q$	2, 4 &I
1, 2	(6)	$\sim P$	3, 5 ~I

Natural Deduction à la Gentzen:

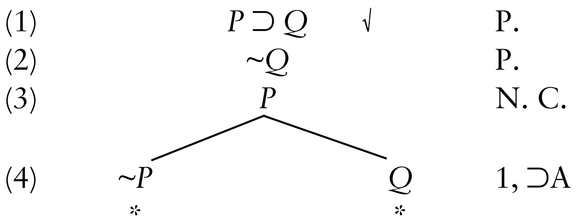


Line of Thought. The last line will be established by ~I. Hence we need to assume *P* and try to derive a contradiction from it. Assuming *P* we can apply ⊃E and derive *Q*. This allows us to use ~*Q*, which is a premise, to derive *Q* & ~*Q* by &I. Given this contradiction, we can derive ~*P* by ~I and discharge our assumption [*P*]. A similar line of reasoning applies to (4) and (6).

Truth Table:

P	Q	$((P \supset Q) \& \sim Q) \supset \sim P$				
		⊃	&	~Q	⊃	~P
T	T	T	F	F	**T**	F
T	F	F	F	T	**T**	F
F	T	T	F	F	**T**	T
F	F	T	T	T	**T**	T

Tree:



(6) *P* & *Q*, ~(*P* & *R*) ⊢ ~*R*

Natural Deduction à la Suppes:

1	(1)	*P* & *Q*	P.
2	(2)	~(*P* & *R*)	P. / ~*R*
3	(3)	*R*	Assumption
1	(4)	*P*	1, &E
1, 3	(5)	*P* & *R*	3, 4, &I
1, 2, 3	(6)	(*P* & *R*) & ~(*P* & *R*)	2, 5, &I
1, 2	(7)	~*R*	3–6, ~I

Natural Deduction à la Gentzen:

$$\dfrac{\dfrac{\dfrac{P \,\&\, Q}{P} \qquad [R]}{P \,\&\, R} \qquad \sim(P \,\&\, R)}{\sim R}$$

Tree:

(1)	*P* & *Q*		√	P.
(2)	~(*P* & *R*)		√	P.
(3)	*R*			N. C.
(4)	*P*			1, &A
(5)	*Q*			1, &A
(6)	~*P* *	~*R* *		2, ~&A