Exploring the ND Rules for ∀I and ∃E

The rules for ∃I and ∀E are straightforward (cf. *Notes and Exercises*, pp. 44–5):

(1) $Fa \vdash \exists xFx$, and
(2) $\forall xFx \vdash Fa$.

It seems unproblematic to infer from a universally generalised sentence to an instance; and it seems unproblematic to infer from an instance to an existantially generalised sentence too.

But the move from $\exists xFx$ to Fa is more tricky, and so is the move from Fa to $\forall xFx$. Unless there are restrictions, it would be permissible to say, e.g., that since Descartes is French, all philosophers are French; or since there is some philosopher who is atheist, Berkeley is an atheist. These deductions are invalid.

First, *Universal Introduction*, ∀I:

(3) $Fa \vdash \forall xFx$, or schematically, in terms of metavariables, $\phi[t/v] \vdash \forall v\phi v$.

Note. 'v' is a metavariable, 't' is a metaconstant (or 'term'), ϕ is a QL meta-wff in which v occurs freely. Thus, '$\phi[t/v]$' is the QL sentence that we get from replacing all free occurrences of v in ϕ by t. For instance, $Rx[a/x]\ \exists x(Rxy \supset \exists yLyx)$ becomes $\exists x(Ray \supset \exists y(Lyx)$. The last occurrence of 'x' is not replaced because it is bound.

There are three conditions on ∀I:

(i) the constant t does not occur in ϕ, and
(ii) the constant t does not occur in any undischarged assumption in the proof of $\phi[t/v]$. In other words, $\phi[t/v]$ does not depend on any line in which t occurs. In short, we must not generalise over a constant that is in the sentence that we try to derive or which figures in an assumtion that is not yet discharged.
(iii) When replacing t with x, the variable x may not be bound by any other quantifier.

Background. While t is a metaconstant, a is a constant of QL with a special role: it stands for an *arbitrary name*. E.g., *Fido* is no particular dog but some general dog, perhaps *the* (representative) dog. Hume discusses a similar point in relation to *abstract ideas*: 'A great philosopher [Berkeley] has disputed the receiv'd opinion on [whether general or abstract ideas are particular or not], and has asserted, that all general ideas are nothing but particular ones, annex'd to a certain term, which gives them a more extensive signification, and makes them recal upon occasion other individuals, which are similar to them" (*Treatise* 1.1.7.1); and "Abstract ideas are therefore in themselves individual, however they may become general in their representation. The image in the mind is only that of a particular object, tho' the application of it in our reasoning be the same, as if it were universal." (*Treatise* 1.1.7.6)

Example. $\forall x(Fx \supset Gx), \forall y(Gy \supset Hy) \vdash \forall x(Fx \supset Hx)$

1	$\forall x(Fx \supset Gx)$	Premise	
2	$\forall y(Gy \supset Hy)$	Premise	
3	$Fa \supset Ga$	1, ∀E	*Note.* 'a' is an arbitrary name.
4	| Fa	Assumption	
5	| Ga	3, 4 ⊃E	
6	| $Ga \supset Ha$	2, ∀E	*Note.* 'a' may be used again.
7	| Ha	5, 6 ⊃E	
8	$Fa \supset Ha$	4–7, ⊃I	*Note.* Assumption *Fa* (4) is discharged.
9	$\forall x(Fx \supset Hx)$	8, ∀I	

Rationale. Remember that a universal statement contains the horseshoe, as in $\forall x(Fx \supset Gx)$, which says that all the *F*s are *G*s. So, the first task is to derive a conditional statement, which we can then universalise. Hence, in the deduction, there will probably be an application of ⊃I somewhere.

Secondly, *Existential Elimination*, ∃E:

(4) $\exists xFx \vdash Fa$, or schematically[1],

$$\dfrac{\begin{matrix} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\phi v & \psi \end{matrix}}{\psi}$$

Restrictions. (i) (i) t does not occur in $\exists v\phi v$; (ii) ψ does not contain t. (iii) ψ is deduced from $\phi[t/v]$ without depending on any line in which t occurs (other than that in which $\phi[t/v]$ itself occurs). That is, t does not occur in any undischarged assumption.

Rationale. In a sense, ∃E works like a conditional proof: if we can deduce ψ from $\phi(t)$, i.e. any *arbitrary* designator, then we have shown that we can deduce ψ from $\exists v\phi v$. The example illustrates this: $\exists xFx, \forall x(Fx \supset Gx) \vdash \exists xGx$.

1	$\exists xFx$	Premise	*E.g.*, There is a rationalist.
2	$\forall x(Fx \supset Gx)$	Premise	*E.g.*, All rationalists like wine.
3	| Fa	Assumption	*Note.* Aim: prove the conclusion and then discharge *Fa* by appying ∃E. We pick an *arbitrary* name and assume it is *F*.
4	| $Fa \supset Ga$	2, ∀E	*E.g.*, If *John Doe* is a rationalist, he likes wine.
5	| Ga	3, 4 ⊃E	
6	| $\exists xGx$	5 ∃I	
7	$\exists xGx$	1, 3–6, ∃E	*Note.* Repeat conclusion to discharge *Fa*.

1 After Halbach V. (2010). *The Logic Manual.* Oxford: Oxford University Press (p. 139).