Exploring the 'Destructive Dilemma'

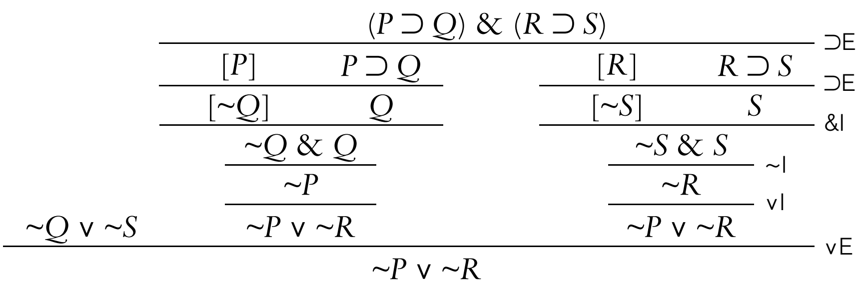
(1) Suppose someone argues like this: "If we read Plato, we become virtuous. And if we read Kant, we become smart. But since we do not become either virtuous or smart, it follows that we do not read either Plato or Kant."

(2) Translated and formalised, we get: $(P \supset Q)\ \&\ (R \supset S), {\sim}Q \lor {\sim}S \vdash {\sim}P \lor {\sim}R$
The structure exemplified here is called a 'destructive dilemma': given two material implications and the disjunction of the two false consequents, we can conclude the disjunction of the two false antecedents. This is a valid structure.

(3) We can prove it with this tree:

(1)	$(P \supset Q)\ \&\ (R \supset S)$	√	Premise
(2)	${\sim}Q \lor {\sim}S$	√	Premise
(3)	${\sim}({\sim}P \lor {\sim}R)$	√	Negated Conclusion
(4)	$P \supset Q$	√	1, ⊃A
(5)	$R \supset S$	√	1, ⊃A
(6)	${\sim}{\sim}P$		3, ~∨A
(7)	${\sim}{\sim}R$		3, ~∨A
(8)	${\sim}P$ * Q		4, ⊃A
(9)	${\sim}R$ * S		5, ⊃A
(10)			
(11)	${\sim}Q$ * ${\sim}S$ *		2, ∨A

(4) We can also prove it in a natural deduction. First, in the more graphical Gentzen style:



(The rules on the right margins are applied on the other branch too.) This proof looks not as straightforward as the tree, because we have to make a range of assumptions.

(5) This becomes more obvious when we present the natural deduction in the more linear Suppes style:

1	(1)	$(P \supset Q)$ & $(R \supset S)$	P.
2	(2)	$\sim Q \vee \sim S$	P. / $\sim P \vee \sim R$
1	(3)	$P \supset Q$	1, &E
1	(4)	$R \supset S$	1, &E
5	(5)	$\sim Q$	A.
6	(6)	P	A.
1, 6	(7)	Q	3, 6 ⊃E
1, 5, 6	(8)	Q & $\sim Q$	5, 7, &I
1, 5	(9)	$\sim P$	5–8, ~I
1, 5	(10)	$\sim P \vee \sim R$	9, ∨I
11	(11)	$\sim S$	A.
12	(12)	R	A.
1, 12	(13)	S	4, 12, ⊃E
1, 11, 12	(14)	S & $\sim S$	11, 13, &I
1, 11	(15)	$\sim R$	11–14, ~I
1, 11	(16)	$\sim P \vee \sim R$	15, ∨I
1, 2	(17)	$\sim P \vee \sim R$	2, 5–16, ∨E

Here is the line of thought. The only thing we can derive easily is line 3 and 4. From these two lines, we need to get $\sim P$ and $\sim R$, since both propositions appear nowhere else. *And* we have to makes use of the second premise, which is line 2. But how can we do this? In two steps, applied twice. We assume P (at line 6), in order to derive Q (at line 7) by *modus ponens* (i.e. ⊃E). But we also first assume $\sim Q$ at line 5. Why? Because this is one of the disjuncts of the second premise, which we use to derive a proposition that allows us to *eliminate* the disjunction. Remember, the rule says that if it is possible to derive some formula χ from ϕ and ψ, which are the two disjuncts of $\phi \vee \psi$, then we can replace the disjunction with χ (see Reader, p. 32). We now use the assumption of $\sim Q$ to derive a contradiction at line 8. This allows us to derive $\sim P$ and discharge the assumption introduced at line 6. By ∨I, we simply add $\sim R$ to it at line 10. Remember: this rule allows us to add *any* disjunct to a proposition. Then we do exactly the same again: assume the other disjunct at line 11, assume R at line 12 to get a contradiction at line 14, and thus its negation at line 15. Then we add the first disjunct of the conclusion at line 16. This looks a little cumbersome, but now we have deduced $\sim P \vee \sim R$ from *both* disjuncts of the second premise. Thus we can write it again on the last line of the proof and thereby discharge the assumptions introduced at lines 5 and 11.[1]

1 I am grateful to James Studd for a critical hint.