Horseshoe and Turnstiles

The *Notes and Exercises* sketch some connections between the horseshoe and the two turnstiles (pp. 19–20, 25). Here are some further details.

ϕ	ψ	$\phi \supset \psi$
T	T	T
T	F	F
F	T	T
F	F	T

The horseshoe '⊃' is a *connective* (or constant) in PL. It is the symbol for the material implication (or material conditional), as in, e.g., '(P & Q) ⊃ Q'. The horseshoe connects two wffs, and it is defined truth-functionally by a specific distribution of truth-values (i.e. T, F, T, T): the formula $\phi \supset \psi$ is false just in case ϕ is true *and* ψ is false, and true in all other cases.

The double turnstile '⊨' is the symbol for the logical implication. It expresses a (binary) *relation* between a premise, or a set of premises (Γ), and a conclusion (ϕ): $\Gamma \vDash \phi$. The relation in question can be called 'following-from relation' or the 'truth-imposing relation', which is why '⊨' is also occasionally called 'semantic consequence'.

Validity can be defined in these terms. An argument A is valid iff there is no evaluation of A's premises under which they are all true yet A's conclusion is false. If A's premises are true, then they impose their truth on the conclusion; or, if A's premises are true, the move to the conclusion preserves truth. Hence, the double turnstile '⊨' is the marker for validity.

So, it is clearly wrong to think of the double turnstile as just a 'stronger' sort of implication, perhaps in the sense of a boxed horseshoe as in '$\Box(\phi \supset \psi)$'.

But how *do* '⊃' and '⊨' relate? By a link with tautology. We can read,

(1) $P \vee Q, {\sim}P \vDash Q$, (more precisely, '$P \vee Q$', '${\sim}P$' ⊨ '$Q$'),

as follows: 'The PL structure with the premises '$P \vee Q$' and '${\sim}P$', and the conclusion 'Q' is a tautology.'

A proposition ϕ is a tautology, or tautologically true, if it is true no matter what, that is, if ϕ's truth is not dependent on any other proposition(s). It is always true. We can express this as,

(2) $\vDash \phi$.

In this case, ϕ is also sometimes called a 'logical truth'. More controversially, we could say that (2) says that ϕ is an axiom or a self-evident truth. Now,

(3) $\Gamma \vDash \phi$ iff there is no truth-evaluation (or assignment of truth-values) of the atomic formulae in Γ and of ϕ that makes the propositions in Γ true and ϕ false.

If we now bring in the idea that '⊃' is defined by the truth table (cf. above), which only excludes the case where the antecedens is true and the consequent false, we can express (3) also as follows,

(4) $\Gamma \vDash \phi$ iff there is no truth-evaluation of the atomic formulae in Γ and of ϕ that makes '$\Gamma \supset \phi$' false.[1]

But this equivalent to saying,

(5) $\Gamma \vDash \phi$ iff every truth-evaluation of the atomic formulae in Γ and ϕ makes '$\Gamma \supset \phi$' true.

Given the idea that a set Γ of PL formulae tautologically entail ϕ means that there is no evaluation of the atomic formulae in Γ and ϕ that makes Γ true but ϕ false, this amounts to saying that,

(6) $\Gamma \vDash \phi$ iff '$\Gamma \supset \phi$' is a tautology.

Given (2), we can now say,

(7) $\Gamma \vDash \phi$ iff $\vDash (\Gamma \supset \phi)$.

In words, the argument from Γ to ϕ is valid if and only if the corresponding material implication $(\Gamma \supset \phi)$ is tautologically true (or a tautology):

P	Q	$(P \vee Q)$	&	$\sim P$	$\supset$	Q
T	T	T	F	F	**T**	T
T	F	T	F	F	**T**	F
F	T	T	T	T	**T**	T
F	F	F	F	T	**T**	F

There is also a connection to the single turnstile '$\vdash$', which expresses a syntactic relation between Γ and ϕ. It says that ϕ can be derived, or proved, from the set of premises. This deducibility relation is due to a system of (sound) inferential rules that connect wffs regardless of what they mean. Hence 'syntactic'. The '$\vdash$' appears in natural deductions and is the hence sign for provability: '$\Gamma \vdash \phi$' means that ϕ is provable from Γ. Unsurprisingly, '$\vdash$' and '$\vDash$' connect:

(8) $\Gamma \vDash \phi$ iff $\Gamma \vdash \phi$.

In words, if ϕ is a logical consequence of, or follows from, Γ, then there is a proof for this entailment. And if there is a proof procedure that derives ϕ from Γ, then Γ tautologically entails ϕ. Suppose this equivalence does not hold. In that case, ϕ could be proved from Γ, yet ϕ would not be entailed by Γ, which in effect means that the argument from Γ to ϕ is invalid even though we can deduce ϕ from Γ. But this makes no sense. In the other case, although the argument from Γ to ϕ would be valid, there would be no proof that reflects that. And this makes no sense either.

1 Strictly, 'Γ' should not be in this formula, but the conjunction of the premises, e.g., 'ψ & χ'.