Predicate Logic III

The Tree Method Sketched.

This is a perhaps more intuitive method than natural deduction to test the validity of arguments (that applies to PL too). One lists the premises and the *negation* of the conclusion and see whether this is consistent. If it is not, the argument is valid; if it is, the argument is invalid.[1] *Example*: $\forall x(Fx \supset Gx)$, $Fa \;\&\; \sim Ha \vdash \sim\forall x(Gx \supset Hx)$, which may stand for, say, 'All rationalists are cyclists; but Leibniz is an unreasonable rationalist; so not all cyclists are reasonable'.

(1)		$\forall x(Fx \supset Gx)$		[Premise]
(2)		$Fa \;\&\; \sim Ha$		[Premise]
(3)		$\sim\sim\forall x(Gx \supset Hx)$		[*Negated* Conclusion]
(4)		Fa		[2, &E]
(5)		$\sim Ha$		[2, &E]
(6)		$\forall x(Gx \supset Hx)$		[3, ~E]
(7)		$Fa \supset Ga$		[1, ∀E]
(8)		$Ga \supset Ha$		[6, ∀E]
(9)	$\sim Fa$ ★	Ga		[7, $\phi \supset \psi$ is true iff ϕ is false *or* ψ is true]
(10)		$\sim Ga$ ★	Ha ★	[8, *do.*]

The '★' indicates an inconsistency; since all 'branches' lead to a contradiction, the argument is valid.

Expanding QL to QL=

Predicate logic with *identity* introduces '=' (and '≠') as new symbols and hence expands the syntax of QL. For instance, $x = y$ says that x and y are one and the same (numerical identity). Gain: more refined propositions, definite descriptions:

(a) 'Some other than Descartes wrote *Hamlet*': $\exists x(Rxa \;\&\; \sim x = b)$

(b) 'There is exactly one logician': $\exists x(Fx \;\&\; \forall y(Fy \supset y = x))$[2]

(c) 'There are at least two philosophers': $\exists x\exists y(Fx \;\&\; Fy) \;\&\; (x \neq y)$

(d) 'There are at most two philosophers': $\forall x\forall y\forall z((Fx \;\&\; Fy \;\&\; Fz) \supset ((x = y \vee x = z) \vee y = z))$

(e) 'The King of France is bald': $\exists x(Rxa \;\&\; \forall y(Rya \supset y = x) \;\&\; Gx)$

(f) 'The King of France does not exist': $\sim\exists x(Fx \;\&\; \forall y(Fy \supset x = y))$[3]

(g) 'Only Socrates is human': $Fa \;\&\; \forall x((x \neq a) \supset \sim Fx)$

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1 For a clear introduction to trees for PL and QL, see, e.g., Smith, P. (2003). *Formal Logic*. Cambridge: Cambridge University Press.

2 Occasionally: $\exists! xFx$.

3 "To be is, purely and simply, to be the value of a variable", Quine, W. V. (1948). On What There Is. *Review of Metaphysics*, *2*, 21–38 (p. 32).

Extensionality.
QL is an extensional language. The extension of a predicate is the set of objects that fall under it. (The extension of a constant, name, or single term is its referent.) In an extensional language, we can substitute expressions with the same extension without altering the extension of the whole sentence: substitution of co-extensive predicates does not change the sentence's truth value (*salva veritate*).

Example. a = George Eliot, b = Maryann Evans, F = is a writer: Fa, $a = b \vdash Fb$.

Natural languages are not extensional. Even though the extensions of 'is a creature with kidneys' and 'is a creature with a heart' are the same (cf. 'centaur' and 'unicorn', or 'morning star' and 'evening star'), the predicates have different meanings: kidneys are not hearts. If a context is sensitive to the mode of presentation, it is *intensional* (i.e. non-extensional). This fact makes for great drama. Oedipus loves Iocasta but he does not *know* (or believe) that Iocasta is (numerically identical with) his mother.

If this situation were transparent, then Loi & $i = m \vdash Lom$. But the referential context is *opaque*. Intentional contexts are intensional. *Modal* contexts are intensional:

(a) Necessarily, every animal with a heart has a heart.
(b) Necessarily, every animal with a heart has kidneys.

We get (b) from (a) by substituting *co-extensional* terms. But (a) is a necessary logical truth (tautology), and (b) is not. Compare:

(c) It is possible that unicorns exist.
(d) It is possible that round squares exist.

Second-Order Predicate Logic.
In QL and QL=, we quantify over individuals. In second-order predicate logic, we quantify over *predicates*, i.e. we treat predicates as variables for properties (Π) and relations (Θ):

(a) $\forall x \exists \Pi \Pi x$ says: for anything there is a predicate that it satisfies; or, everything has some property or other.
(b) $\exists \Pi \Pi a$ says: there is some feature and an individual has it.
(c) $\exists \Pi \forall x \Pi x$ says: there is some feature that everything has.
(d) $\forall x \forall y(x = y \supset \forall \Pi(\Pi x \equiv \Pi y))$ says: identical objects have the same properties (i.e. indiscernibility of identicals).
(e) $\forall x \forall y(\forall \Pi(\Pi x \equiv \Pi y) \supset x = y)$ says: any two objects that have the same properties are (numerically) identical (i.e. identity of indiscernibles). This is known as 'Leibniz's Law'.[4]
(f) $\forall \Theta \exists \Pi \forall x \forall y(\Theta xy \supset (\Pi x \; \& \; \Pi y))$ says: For any relation there is a feature such any two things that stand in this relation have a that feature.

4 As Leibniz says, 'no two substances are entirely alike, and only differ in number' (*Discourse on Metaphysics*, §9).