Clarifying the Rules for Trees

This argument created a puzzle:

‘~(*P* ∨ *Q*)’, ‘*P* & *R*’ ⊧ ‘~(*R* ∨ *Q*)’

In the tree we could not close one branch even though the argument is valid, as this truth table shows:

P	*Q*	*R*	(~	(*P* ∨ *Q*)	&	(*P* & *R*))	⊃	~	(*R* ∨ *Q*)
T	T	T	F	T	F	T	**T**	F	T
T	T	F	F	T	F	F	**T**	F	T
T	F	T	F	T	F	T	**T**	F	T
T	F	F	F	T	F	F	**T**	T	F
F	T	T	F	T	F	F	**T**	F	T
F	T	F	F	T	F	F	**T**	F	T
F	F	T	T	F	F	F	**T**	F	T
F	F	F	T	F	F	F	**T**	T	F

Lines 1–3 and 5–7 have false conclusions, but in all these cases the conjunction of the premises is false too. Since a material implication with a false antecedent is true, the premises tautologically entail the conclusion. This proves validity.

But the tree is not so conclusive. We can start thus:

(1)	~(*P* ∨ *Q*)	√	Premise
(2)	*P* & *R*	√	Premise
(3)	~~(*R* ∨ *Q*)		Negated Conclusion
(4)	*R* ∨ *Q*		3, ~~A
(5)	~*P*		1, ~∨A
(6)	~*Q*		1, ~∨A
(7)	*P*		2, &A
(8)	*R*		2, &A

Since this leaves line 3 unanalysed, we cannot stop there, even if the trunk *closes* because of line 5 and 7. But if we continue with the tree, we get an open branch:

(9)	*R*	*Q*	4, ∨A
	○	*	

So, it looks as it our valid argument cannot be shown to be valid in a tree!

Either we analyse every wff of the argument or not. If we do, we cannot demonstrate the argument’s validity, and if we do not, we close off the tree too early. Given this (apparent) dilemma, let us try another decomposition:

(1*)	$\sim(P \vee Q)$			√	Premise
(2*)	$P \& R$			√	Premise
(3*)	$\sim\sim(R \vee Q)$			√	Negated Conclusion
(4*)	$R \vee Q$				3*, ~~A
(5*)		R	Q		4*, ∨A
(6*)		$\sim P$	$\sim P$		1*, ~∨A
(7*)		$\sim Q$	$\sim Q$		1*, ~∨A
(8*)		P	P		2*, &A
(9*)		R	R		2*, &A

There are two inconsistencies on each branch, which is good news. But *R* is still not involved, as it were. So, can we close off the branches and declare the argument valid anyway? Yes.

Here is why. The key idea is that open branches indicate *counterexamples* that thus falsify our assumption. Here is a simple example: '$P \supset Q$', 'Q' $\vDash$ 'P'

(1)	$P \supset Q$			√	Premise
(2)	Q				Premise
(3)	$\sim P$				Negated Conclusion
(4)		$\sim P$ ○	Q ○		1, ⊃A

On the open branches, ~*P* is T, so *P* is **F**, and *Q* is **T**. This particular distribution of truth values corresponds to the following line in the truth table,

P	Q	$P \supset R$	Q	P
F	T	T	T	F

which shows that the premises are true *but* the conclusion is false. Hence the argument is invalid.

In our puzzling case, *R* could not generate a counterexample, because all other variables are already inconsistent. That is, while we know that *R* is **T**, it is not possible to assign a truth value to *P* and *Q*, and hence it is impossible to construct a counterexample. There are similar cases where variables appear in the tree that are not inconsistent, yet *others* on the branch are. And this is enough to close it.

(a) $(P \& Q) \vee \sim(P \supset Q) \vDash R \supset P$
(b) $(P \vee (Q \vee R), P \supset Q, Q \equiv S \vDash Q \& S$ [1]
(c) $\vDash (P \vee (Q \& R)) \supset ((P \vee Q) \& (P \vee R))$[2]

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[1] After Smith, N. J. J. (2012). *Logic: The Laws of Truth*. Princeton: Princeton University Press (pp. 150, 154).

[2] From Smullyan, R. M. (1968). *First-Order Logic*. New York: Dover Publications (p. 16).